Math 206A Lecture 2 Notes

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1 Helly's Theorem

1.1 Proof of Helly's theorem

Theorem 1.1 (Helly). *Suppose $X_1, \dots, X_n \subseteq \mathbb{R}^d$ are convex sets such that $X_I \neq \varnothing$ for all $|I| = d+1$, where $X_I = \bigcap_{i \in I} X_i$. Then $X_1 \cap \cdots \cap X_n \neq \varnothing$.*

When $d = 1$, we have a collection of intervals where every pair of intervals intersect; then all intervals intersect. In this case the proof is elementary. Take the largest left endpoint a^* and the smallest right endpoint b^* of one of the intervals. Then $a^* < b^*$, so a point between a^* and b^* is contained in all the intervals.

However, when $d = 2$, the result is a little less obvious.

Proof. Let's prove the theorem for $d = 2$, $n = 4$. Let $J = \{1, \dots, n\}$. Let $y_i \in X_{J \setminus \{i\}}$. Either one of the y_i lies in the triangle formed by the three others or the y_i form a convex shape. In the first case, without loss of generality, $y_4 \in X_1 \cap X_2 \cap X_3$. But if $y_1, y_2, y_3 \in X_4$, then $y_4 \in X_4$. In the second case, find the point z at the intersection of the line segments connecting y_1 to y_3 and y_2 to y_4. Then $z \in X_2 \cap X_4$, and $z \in X_1 \cap X_3$. So $z \in X_J$.

Now proceed by induction on n. Why does n imply $n+1$? The proof is the same, except we just include the points $y_i \in X_5, X_6, \dots$. So in the first case, we just ignore the extra points, we get

$$z \in (X_2 \cap X_4 \cap X_5 \cap \cdots \cap X_{n+1}) \cap (X_1 \cap X_3 \cap X_5 \cap \cdots \cap X_{n+1}) = X_J$$

for the second case.

Before we prove the general case, we will state a lemma. $\square$

Lemma 1.1 (Radon). *Let $y_1, \dots, y_m \in \mathbb{R}^d$, where $m \geq d+2$. Then there exist $I, I' \neq \varnothing$ such that $I \cap I' = \varnothing$ and the convex hull of $\{y_i : i \in I\}$ intersects the convex hull of $\{y_j : j \in I'\}$.*

Proof. Let $y_i = (y_{i_1}, \dots, y_{i,d}) \in \mathbb{R}^d$ with $i = 1, \dots, m$, $m \geq d+2$. Consider the system of equations $\sum_{i=1}^m \tau_i = 0$ and $\sum_{i=1}^m \tau_i y_{i,j} = 0$ for $j \in \{1, \dots, d\}$. These are $d+1$ equations. So there exist $(\tau_1, \dots, \tau_m) \neq 0$ which satisfies the system. Let $I = \{i : \tau_i > 0\}$ and $I' = \{i : \tau_i < 0\}$. Then

$$\frac{\sum_{i \in I} \tau_i y_i}{c} = \frac{\sum_{j \in I'} (-\tau_j) y_j}{c},$$

where $c = \sum_{i \in I} \tau_i = \sum_{j \in I'} -\tau_j$. $\square$

Now we can prove the general case of Helly's theorem.

Proof. For general d, we induct on n. The base case is $n = d+1$. By the lemma, we get $z \in X_r$, where $r \notin I$, and $z \in X_s$, where $s \notin I'$. So $z \in X_J$. $\square$

1.2 Applications of Helly's theorem

Corollary 1.1. *Let $R_1, \dots, R_n \subseteq \mathbb{R}^2$ be axis-parallel rectangles. Suppose $R_i \cap R_j \neq \varnothing$ for all i, j. Then $R_1 \cap \cdots \cap R_n \neq \varnothing$.*

We could have proved this like we proved the case of $d = 1$ because the intersection of rectangles is the pair of intersections of the corresponding intervals.

Corollary 1.2. *Let $A \subseteq \mathbb{R}^2$ be a fixed convex set, and let $X_1, \dots, X_n \subseteq \mathbb{R}^2$ be convex sets such that for $|I| = 3$, there exists some $c \in \mathbb{R}^2$ such that $X_i \cap (A + c) \neq \varnothing$ for all $i \in I$. Then there exists some $c \in \mathbb{R}^2$ such that $X_i \cap (A + c) \neq \varnothing$ for all $i \in \{1, \dots, n\}$.*

This says that if there is some translation where A intersects some of the X_i there is some translation where A intersects all of them.

Proof. Pick some point in $a \in A$, and look at all A translated by the extreme points of X_i. Let $\hat{X}_i$ be the convex hull of the translated copies of a. Then $\hat{X}_i$ is convex, so $\hat{X}_I \neq \varnothing$ for all $|I| = 3$. By Helly's theorem, $\hat{X}_J \neq \varnothing$, which completes the proof. $\square$

Remark 1.1. If we take A to be a point, we get the original statement of Helly's theorem for $d = 2$.